PAPER CODE NO. **PHYS393**	EXAMINER: DEPARTMENT:	Dr. K. M. Hock Physics	TEL. NO. 01925-603853

JANUARY EXAMINATIONS 2013

Bachelor of Science: Year 3
Master of Physics: Year 3
Master of Physics: Year 4

STATISTICAL AND LOW TEMPERATURE PHYSICS

TIME ALLOWED: 3 hours

INSTRUCTIONS TO CANDIDATES

Answer all questions.

Question 1 carries 50% of the total marks.

Questions 2 and 3 each carry 25% of the total marks.

Answer either part (a) or part (b) of questions 2 and 3.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

The marks allotted to each part of a question are indicated in square brackets.

All symbols have their usual meanings unless otherwise stated.

Question 1.

(a) Cerium Magnesium Nitrate (CMN) is a spin ½ salt. One mole of CMN at 1 K is placed in magnetic field of 1 T.

i) What are the energies of the magnetic energy levels? **[2]**

ii) What is the name for the distribution of the spin ½ ions? State the formula for this distribution. Explain the symbols used. **[2]**

iii) State the formula for the Boltzmann factor. Calculate it for each energy level. **[2]**

iv) Starting from the formula for the distribution, explain how to find the ratio of the number of spin ½ ions in the lower level to the number in the higher level using the answers from (iii). Write down this ratio. **[2]**

v) Using this ratio, find the number of moles of spin ½ ions in each energy level. **[2]**

- -

Solution

1(a)

(i) The energies are $-\mu_B B$ and $+\mu_B B$.

$\mu_B B = 9.27 \times 10^{-24} \times 1 = 9.270 \times 10^{-24}$ J [U2]

(ii) Boltzmann distribution. $n_i = A \exp(-\varepsilon_i/k_B T)$

i label for each level

n_i number of particles in level i

ε_i energy of level i

T temperature

k_B Boltzmann constant

A unknown constant

[U2]

(iii) Boltzmann factor is $\exp(-\varepsilon_i/k_B T)$

$\mu_B B/k_B T = 9.27 \times 10^{-24} / 1.38 \times 10^{-23} = 0.6717$

Lower state: $\exp(+\mu_B B/k_B T) = 1.9576$

Higher state: $\exp(-\mu_B B/k_B T) = 0.5108$ [U2]

(iv)

Populations in the two levels are

$n_1 = A \exp(-\varepsilon_1/k_B T)$

$n_2 = A \exp(-\varepsilon_2/k_B T)$

Dividing gives

$n_1 : n_2 = \exp(+\mu_B B/k_B T) : \exp(-\mu_B B/k_B T) = 3.8324 : 1$ [U2]

(v)

Population in lower state = $n_1 / (n_1 + n_2)$ x 1 mole = 0.7931 mol

Population in higher state = $n_2 / (n_1 + n_2)$ x 1 mole = 0.2069 mol [U2]

- -

(b) A cubic box contains 1 mole of neon gas (relative atomic mass = 20) at 300 K.

i) State the formula for the average kinetic energy of the neon atoms. Calculate this energy. **[2]**

ii) State the formula relating energy and wavevector. Find the wavevector that corresponds to the average kinetic energy of the neon atoms. **[3]**

iii) Find the spacing between wavevectors. The side of the cubic box is 30 cm long. **[2]**

iv) The total energy can be calculated by adding up the energies of all the atoms directly. Why do we still need density of states? **[3]**

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Solution

1(b)

(i) Assuming that neon is an ideal gas, the formula is $3k_B T/2$

the average energy is $3k_B T/2 = 6.210 \times 10^{-21}$ J [U2]

(ii) $$E = \frac{\hbar^2 k^2}{2m}$$

E is the average energy above.

k is wavevector

m is the mass of Neon atom = $20m_u$

$$k = \frac{\sqrt{2mE}}{\hbar} = 1.934 \times 10^{11}\ \text{m}^{-1}.$$ [U3]

(iii) The spacing is $\pi/L = \pi/0.3 = 10.47\ \text{m}^{-1}$. [U2]

(iv)

Density of states help us to find the sums for number of particles and total energy when the number of energy levels is very large.

This happens when the average energy of the particles is large compared with the spacing between levels. [U3]

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(c) There is some spin ½ salt in a magnetic field. At high temperatures, there are 0.1 mole of the spin ½ ions at each of the two magnetic energy levels.

i) When temperature is lowered to 1 K, the number of the ions at the lower level increases by 50%. Find the amount of spin ½ ions at each level in moles. **[3]**

ii) Using this result, calculate the difference in energy between the two energy levels. **[3]**

iii) Hence, calculate the amount heat is given out by the salt when temperature is lowered to 1 K. **[4]**

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<u>Solution</u>

1(c)

(i) At high T,

Population at lower level = 0.1 mol

Population at higher level = 0.1 mol

At 1 K, Population at lower level $n_1 = 0.1 + 0.05 = 0.15$ mol

So, population at lower level $n_2 = 0.1 - 0.05 = 0.05$ mol [U3]

(ii)

Let the energies of the levels be ε_1 and ε_2.

The macrostate is Boltzmann distribution.

So $n_1 = 0.15 \text{ mol} = \exp(-\varepsilon_1/k_BT)$

and $n_1 = 0.05 \text{ mol} = \exp(-\varepsilon_2/k_BT)$

Dividing gives $3 = \exp((\varepsilon_2 - \varepsilon_1)/k_BT)$

Solving, the energy difference is $\varepsilon_2 - \varepsilon_1 = k_BT \ln 3 = 1.516 \times 10^{-23}$ J [U3]

(iii)

From above, 0.05 mol of ions moved from higher to lower energy.

This is a number of $0.05\ N_A$. [U2]

The energy of each ion falls by $\varepsilon_2 - \varepsilon_1$. This is given out as heat.

So the total heat given out is $0.05\ N_A (\varepsilon_2 - \varepsilon_1) = 0.4563$ J [U2]

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(d)

i) Why is the Faraday's law of electromagnetic induction not able to explain Meissner's effect? **[2]**

ii) The presence of a magnetic field in a macroscopic wavefunction of electrons must produce a current. Why is this able explain the Meissner's effect? **[2]**

iii) In the Meissner's effect, some magnetic field remains in the superconductor. Why is the field not completely expelled? What is the name given to the depth of penetration? **[3]**

iv) Sketch the heat capacity versus temperature graph for a superconductor, above and below the transition temperature. On the same graph, sketch the heat capacity if a magnetic field destroys the superconductivity. How do these graphs suggest the existence of an energy gap? **[3]**

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<u>Solution</u>

1(d)

(i)

Suppose a magnetic field is already present above transition temperature. When the body becomes superconducting, a current would still appear that expels the field. This current appears even though there is no change in flux, which is not what Faraday's law says.

[B2]

(ii)

When the body becomes superconducting, a macroscopic wavefunction appears. This wavefunction produces a current when there is a field. This current can then expel the field. So the changing flux in Faraday's law is not needed.

[B2]

(ii)

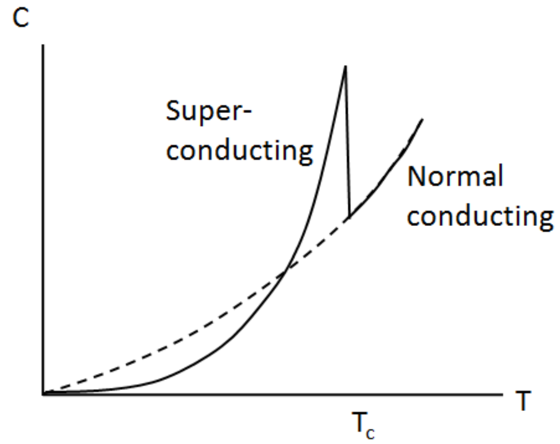
The reason is that some current is needed to produce the opposing field.

This current is produced by the field.

In order for current to exist, some field must penetrate into the superconductor. [B2]

The depth of this penetration is the London's penetration depth. [B1]

iii)



[B2]

Above transition, the heat capacity follows that of the normal metal, $c_v = AT + \gamma T^3$.

Below transition, it changes to $c_v = B \exp(-\Delta/k_B T)$, which has the same form as a Boltzmann factor between two levels of energy gap Δ. [B1]

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(e) A body can move with zero resistance through superfluid ^{4}He.

i) What excitations are possible in superfluid ^{4}He? **[2]**

ii) Write down the dispersion relation for a particle in free space. With the help of a graph, obtain the critical velocity of a body moving through an ideal gas. **[2]**

iii) Sketch the dispersion curve of superfluid ^{4}He. Explain the main features and how to estimate the critical velocity. **[4]**

iv) Why does the body experience no resistance when its velocity is below the minimum E/p? **[2]**

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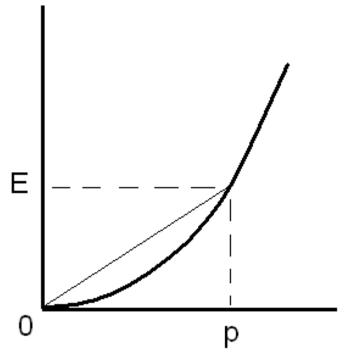
Solution

1(e)

(i) Phonons.

rotons. [B2]

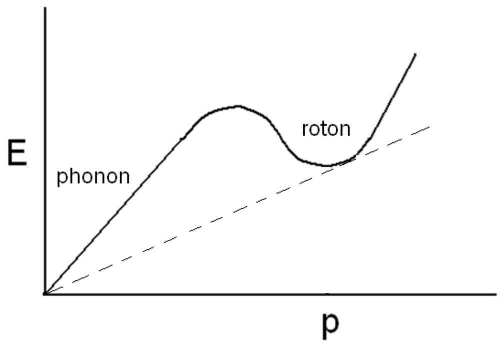
(ii) Free particle dispersion relation is $E = p^2/2m$.



Formula $v_c = (E/p)_{min}$. At p = 0, straight line is horizontal.
So minimum E/p is 0. So Critical velocity = 0.

[B2]

(iii)



For small p, phonons are excited, and dispersion is straight line.
For larger p, rotons are excited, and dispersion shows a minimum.

[B2]

Critical velocity $v_c = (E/p)_{min}$.
Is obtained by drawing straight line from origin to touch the curve near the minimum at higher p.
This has the smallest gradient of E/p, which gives the critical velocity.

[B2]

(iv)

For velocity below the minimum E/p, no excitation is possible.

If there is no excitation, the body cannot lose energy, so it experiences zero resistance. [B2]

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Question 2. Answer **<u>either</u>** (a) **<u>or</u>** (b)

2(a)

In copper metal, each atom provides one conduction electron.

i) State the formula for the average kinetic energy of the conduction electrons, assuming that they behave like an ideal gas. Find the heat capacity of one mole of copper due to these conduction electrons. **[3]**

ii) The molar heat capacity of copper is found to be 0.6 mJ/K at 1 K. Why is this so much smaller than the answer in (i)? **[2]**

iii) If temperature is not too high, only the electrons that are just below the Fermi level are excited. Using a graph, estimate the energy interval below Fermi level from which electrons are excited. Give the answer in terms of the temperature, T. **[4]**

iv) Following on from (iii) and given that the density of states for the ideal gas is

$$g(\varepsilon) = \frac{4m\pi V}{h^3}\sqrt{2m\varepsilon}\,,$$

where V is the volume, derive an expression for the number, n, of excited electrons, in terms of Fermi energy E_F. **[4]**

v) Using a graph, explain what is the approximate distribution of these excited electrons considered in (iii) and (iv). Why do these excited electrons seem tp behave like an ideal gas? **[4]**

vi) The ideal gas formula for heat capacity is given by the answer to (i). Using this and the answer to (iv), derive an expression for the heat capacity due to these excited electrons. **[4]**

vii) Find the Fermi energy

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3},$$

where N is the number of electrons and V is the volume. (Molar volume of copper is 7.11 cm^3.)**[2]**

viii) Using the answer to (vi), find the heat capacity at 1 K. Compare with values in (i) and (ii) and comment. **[2]**

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<u>Solution</u>

2(a)

(i) The average kinetic energy is $\varepsilon = 3k_BT/2$. [U1]

There is 1 conduction electron per atom, so there are N_A conduction electrons in total.

The total energy $U = N_A\varepsilon = N_A \times 3k_BT/2 = 3RT/2$.

The heat capacity $C = dU/dT = 3R/2 = 12.47$ J/K [U2]

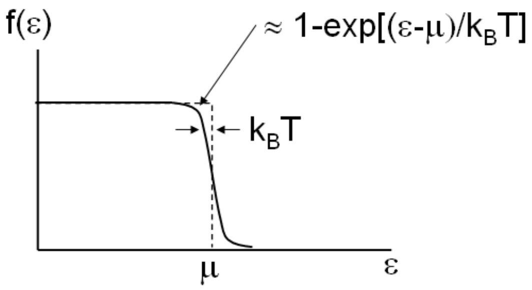
(ii) The electrons do not behave like ideal gas at all.

In the electron gas, the electrons are stacked up in energy levels, from the ground state to a maximum energy. When heated, only a small fraction of the electrons near the top can be excited. As a result, the heat capacity is much smaller. [U2]

(iii) The probability that a state at energy ε is given by the Fermi-Dirac distribution

$$f(\varepsilon) = 1 / (\exp[(\varepsilon - \mu)/(k_BT)] + 1).$$

At 1 K, the graph is quite close to the step at the Fermi energy $\mu = E_F$.



This is because the Fermi energy for a metal is usually much higher than k_BT. [U2]

So for energy a few k_BT smaller than μ, the exponential function quickly becomes small. Then

$$f(\varepsilon) \approx 1 - \exp[(\varepsilon - \mu)/(k_BT)].$$

When energy falls below μ, the probability moves towards 1 exponentially. It falls roughly to 1/e after k_BT.

So k_BT is approximately the range of energy of the electrons that are excited. [U2]

(iv) The density of states is $2 \times g(\varepsilon) = 2 \times (4m\pi V/h^3)(2m\varepsilon)^{1/2}$.

At the Fermi energy, this is $2g(E_F)$. [U2]

The number of excited electrons is the number of states in their energy range, which is k_BT.

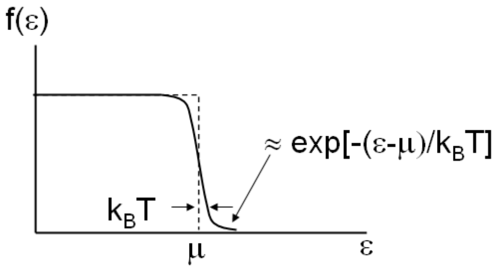
This number, n, of the states is $2g(E_F) \times k_BT$. [U2]

(v) For energy above E_F, the exponential term in

$$f(\varepsilon) = 1 / (\exp[(\varepsilon - \mu)/(k_BT)] + 1).$$

quickly gets large, so that

$f(\varepsilon) \approx \exp[-(\varepsilon - \mu)/(k_BT)]$. [U2]



This is just the Boltzmann distribution, the same as that of an ideal gas. [U2]

(vi) The average energy of an ideal gas particle is $3k_BT/2$.

Since the excited electrons behave like an ideal gas, the total energy is $U = n \times 3k_BT/2$. [U2]

Using the previous expression for n, $U = 2g(E_F) \times k_BT \times 3k_BT/2$

$$= 3g(E_F)k_B^2T^2$$

Heat capacity $C = dU/dT = 6g(E_F)k_B^2T$ [U2]

(vii) $N = N_A$. V = molar volume. m = mass of electron

The Fermi energy is $E_F = (\hbar^2/2m)(3\pi^2N/V)^{2/3} = 1.117 \times 10^{-18}$ J.

[U2]

(viii)

Substituting, we find $C = 6g(E_F)k_B^2T = 0.4552$ mJ/K.

This is much closer to the measured 0.6 mJ/K, compared to the ideal gas result of 12.47 J/K.

This is strong evidence that the Fermi-Dirac distribution is correct. [U2]

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2 (b)

Dilution cooling makes use of a solution of liquid ^3He in liquid ^4He.

i) Sketch a phase diagram of the mixture. Label the axes, the lambda line and the phase separation region. **[5]**

ii) Describe qualitatively what happens when the temperature of a 50% mixture falls just below the phase separation curve. **[5]**

iii) Describe qualitatively what happens when the temperature of this mixture reaches a few milliKelvin. **[3]**

iv) Sketch a diagram to explain qualitatively how the ^3He could be removed from the bottom layer. **[6]**

v) Why is it necessary to remove ^3He from the bottom layer? How does this affect the amount of ^3He in the top layer? What must be done to maintain the same amount of ^3He in the top layer? **[6]**

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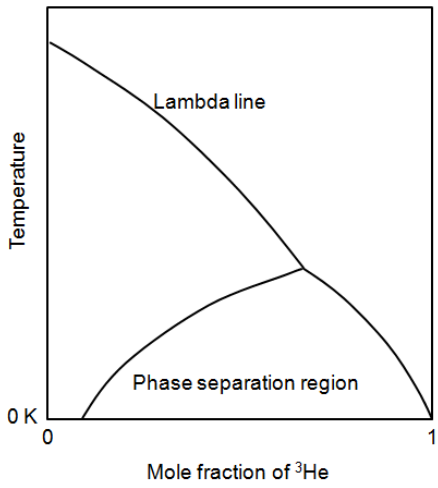
Solution

2(b)

(i)



For graph [B2]

For labels [B3]

(ii)

At a temperature in the phase separation region, the concentration of ^3He that is in the region is not possible.

Instead, the mixture would separate into two layers. [B2]

One layer has a higher concentration of ^{3}He than the other layer.

The layer with higher concentration of ^{3}He is on top, because 3He atom is lighter than 4He atom.

[B3]

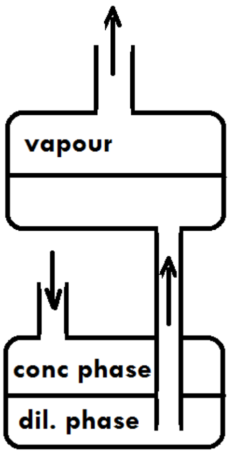
(iii)

The mixture separates into two layers.

The top layer is nearly pure ^{3}He.

The bottom layer reaches a small but nonzero fraction of ^{3}He in the mixture. [B3]

(iv)



[B3]

A tube leads from the bottom layer to a chamber higher up.

Pressure on the top layer forces the bottom layer up to the higher chamber.

There, the dilute phase vaporises.

The vapour contains a higher concentration of ^{3}He, because

^{3}He has higher vapours pressure.

In this way, ^{3}He is removed from the bottom layer. [B3]

(v)

Cooling stops when the dilute phase is saturated with ^{3}He at 6.6%.

^{3}He has to be removed to allow more ^{3}He to go from top to bottom layer,

so that cooling can continue. [B3]

The volume of the top layer would decrease as ^{3}He moves to the bottom layer.

^{3}He must be replaced.

This is done by liquefying the ^{3}He that has been removed from the higher chamber,

and passing it back down to the top layer. [B3]

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Question 3. Answer **either** (a) **or** (b)

(a)

i) Discuss the main features of Bose-Einstein condensate. Suggest why it is a good candidate for explaining superfluidity. **[5]**

ii) The chemical potential μ of a boson gas changes as temperature falls to 0 K. Explain how it changes, using the Bose-Einstein distribution graph. **[5]**

iii) Write down the formula for the number of bosons N in terms of density of states $g(\varepsilon)$, where ε is energy of a particle. When temperature T falls below a certain value, μ has to be set to zero. Sketch the graph of N against T and explain the main features. **[5]**

iv) When μ is zero, the result of the integral in (iii) is

$$N_{ex} = 2.612V\left(\frac{2\pi mk_B T}{h^2}\right)^{3/2},$$

where V is the volume that contains the N particles. Explain how to find the condensation temperature, T_{BE}. Find T_{BE} for liquid ^{4}He (molar volume 27.58 cm^3). **[5]**

v) The total energy U below T_{BE} is given by

$$U = 0.7704k_B N\frac{T^{5/2}}{T_{BE}^{\ 3/2}}.$$

Derive the heat capacity, C, and find it for one mole of ^{4}He at T_{BE}. Sketch the graph of C versus T above and below T_{BE}. Give the answer in units of the gas constant R. Explain the main features.

[5]

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Solution

3(a)

(i)

BEC is a macroscopic wavefunction. The number of particles condensed to the ground state is a significant fraction of the bulk total.

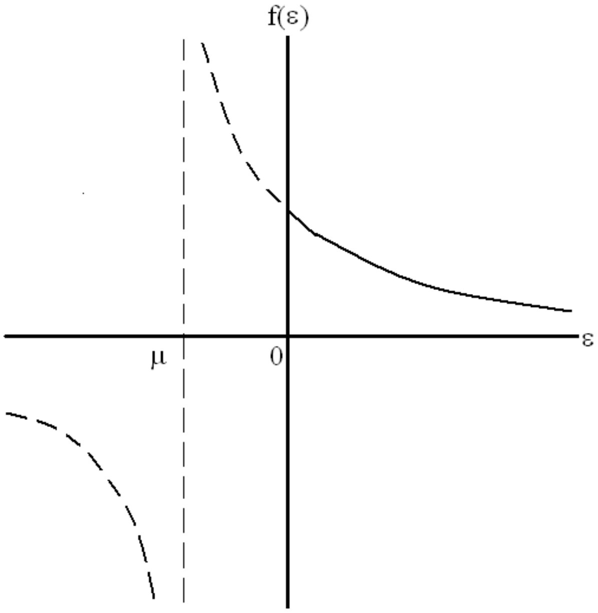
The wavefunction is in one energy state. The next higher state is separated by some amount of energy. If there is not enough energy, the particles cannot get excited. [B2]

Atoms in a superfluid flow round an obstacle without any viscosity. Electrons in a wavefunction move without losing energy.

This is possible if the atoms or electrons form a macroscopic wavefunction. If the flow is slow, there is not enough energy to excite the particle. Then there is no energy loss, so no resistance.[B2]

Since the BEC is a macroscopic wavefunction, this makes it a good candidate. [B1]

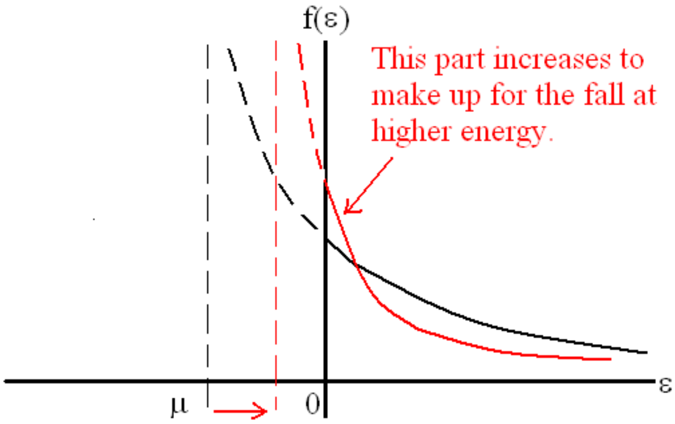
(ii)



Energy is positive. So $\mu < 0$, else $f(\varepsilon)$ can be negative. [B2]

As $T \to 0$, $f(\varepsilon) = 1/[\exp((\varepsilon-\mu)/k_BT)-1]$ falls. [B1]

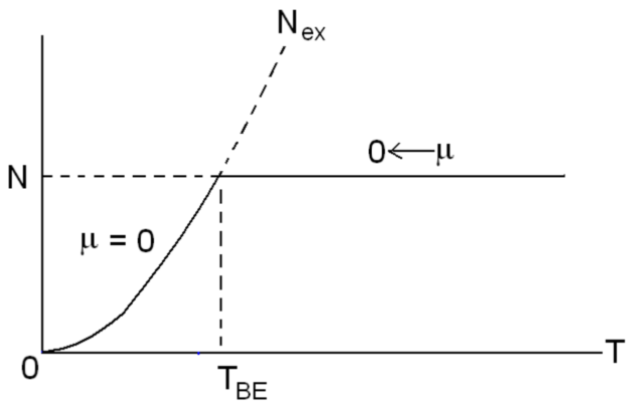
So that $N = \int g(\varepsilon)f(\varepsilon)d\varepsilon$ remains the same, μ must $\to 0$



[B2]

(iii)

$$N = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon-\mu)/k_BT)-1}$$ [U1]



When T falls, $\mu \rightarrow 0$ so that integral remain = N. [U2]

When $\mu = 0$, T is defined as T_{BE}, the condensation temperature.

μ cannot become positive, so it must be fixed at 0. So the integral becomes

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon / k_B T) - 1}.$$

If T falls below T_{BE}, then integral becomes < N.

Interpretation: The decrease in N is because particles condensed to ground state. So N_{ex} is the number that remain excited electrons. [U2]

(iv)

At T_{BE}, the particles just start condensing.

So N_{ex} = N the total. [U2]

So can find T_{BE} using

$$N = \left(\frac{2\pi m k_B T_{BE}}{h^2} \right)^{3/2} 2.612V$$ [U1]

Use N = N_A

V = 27.58 cm^3 = 27.58 x 10^{-6} m^6.

M = 4*m_u.

solve for T_{BE}, get 3.144 K. [U2]

(v)

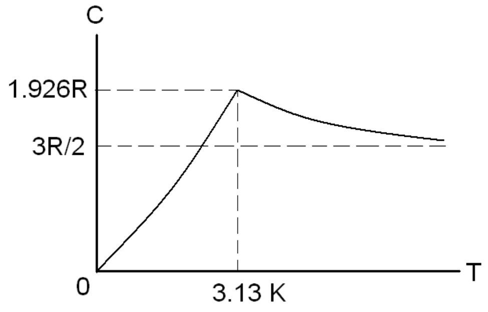
C = dU/dT = $0.7704 k_B N \dfrac{(5/2)T^{3/2}}{T_{BE}^{3/2}}$

Use $N = N_A$, $T = T_{BE}$.

Find $C = 0.7704 k_B N_A (5/2) = 1.926\ R$. [U2]

$T \rightarrow 0$, $C \rightarrow 0$.

$T \rightarrow$ large, $C \rightarrow$ ideal gas result $3R/2$.



[U3]

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3 (b)

In a superconducting metal, the electrons form Cooper pairs. A Cooper pair is a pair of electrons which attract each other.

i) This means that some energy is needed to separate them. Describe one experimental result which suggests the existence of such a binding energy. **[5]**

ii) The superconducting property depends on vibration of the atoms. Describe one experiment which demonstrates this. **[5]**

iii) With the help of a picture, explain qualitatively how movements of the atoms could cause two electrons to attract. **[5]**

iv) The binding energy in the Cooper pair is expected to be much smaller than the kinetic energy of the electrons. Using the Fermi level, explain why the two electrons do not separate from each other. **[5]**

v) A macroscopic wavefunction has been used to explain the Meissner's effect. What is wrong with this idea? How does forming of Cooper pairs help to solve this problem? **[5]**

- -

Solution

3(b)

(i)

One experiment is measurement of heat capacity of a superconductor.

A graph of heat capacity against temperature shows a sudden jump at the transition temperature.

Above transition temperature, heat capacity comes from electrons and phonons. [B2]

Below, it follows exp($-\Delta/k_B T$)

This is like Boltzmann distribution for two energy levels.

So a binding energy suggests two levels – one for bound state, one when the bond is broken. [B3]

(ii) One experiment is the measurement of transition temperature of mercury.

The results show that different isotopes of mercury have different transition temperature.

The difference between isotopes is the masses of the atoms and the number of neutrons.

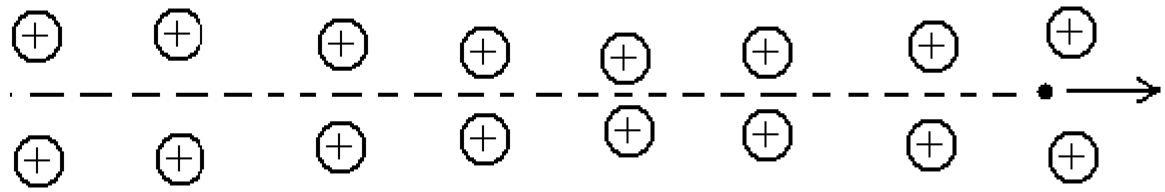
The neutrons do not interact directly with the electrons. [B2]

Movements of atoms do. We know this from effect of heat on conduction.

Heavier atoms move more slowly.

This suggests lattice vibration is an effect on superconductivity. [B3]

(iii)



[B3]

When a conduction electron moves in between the positive ions, it attracts them.

So the ions come closer.

This produces a region of higher positive charge.

This region in turn attracts other electrons. [B2]

(iv)

The two electrons in the Copper pair come from the Fermi level.

When they form the Cooper pair, their energies decrease slightly.

This decrease is the binding energy from the attraction.

This means that the energy of each electron in the pair is now slightly lower than the Fermi energy.

We would expect the electrons to escape because their kinetic energies are still much larger than the binding energy. [B2]

However, if they just come apart, their energies would still be below the Fermi level, where all states are already occupied.

This is not allowed by the exclusion principle.

So if the electrons escape from the Cooper pair, they have no place (energy states) to go to.

By staying together, they form a boson, which does not have to obey the exclusion principle.

In this way, they are forced to stay together. [B3]

(v)

Macroscopic wavefunction means many electrons all occupy the ground state. [B1]

Electrons obey exclusion principle.

So they cannot all occupy the same ground state to form a macroscopic wavefunction.

This is where with the idea is wrong. [B2]

The Cooper pair is a boson.

So they can all occupy the same state to form a macroscopic wavefunction.

This solves the problem that electrons cannot occupy the same ground state. [B2]

- -

CONSTANTS

Speed of light in vacuum	c	=	3.00×10^{8} ms^{-1}
Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$ Hm^{-1}
		=	$4\pi \times 10^{-7}$ VsA^{-1}m^{-1}
Permittivity of vacuum	ε_0	=	8.85×10^{-12} Fm^{-1}
		=	8.85×10^{-12} AsV^{-1}m^{-1}
Elementary charge	e	=	1.60×10^{-19} C
Planck constant	h	=	6.63×10^{-34} Js
	$h/2\pi = \hbar$	=	1.05×10^{-34} Js
Avogadro constant	N_A	=	6.02×10^{23} mol^{-1}
Boltzmann constant	k_B	=	1.38×10^{-23} JK^{-1}
Gas constant	R	=	8.31 JK^{-1}mol^{-1}
Unified atomic mass constant	m_u	=	1.66×10^{-27} kg
		=	931.5 MeVc^{-2}
Electron mass	m_e	=	9.11×10^{-31} kg
Proton mass	m_p	=	1.67×10^{-27} kg
Gravitational constant	G	=	6.67×10^{-11} Nm^{2}kg^{-2}
Acceleration due to gravity	g	=	9.8 ms^{-2}
Bohr magneton	μ_B	=	9.27×10^{-24} JT^{-1}